which is the required formula.[1]

(b) Since a and d are positive acute angles, the last term cannot become negative and is zero only when $x = 0$. Therefore d will have a minimum value when $x = 0$, so that $r = r' = \frac{1}{2}a$, and $i = i'$, and the complete ray is symmetrically situated with respect to the prism.

(c) For this minimum value of d, say d_0, we have the classical laboratory formula

$$n = \frac{\sin \frac{1}{2}(a + d_0)}{\sin \frac{1}{2}a}$$

2879 [1921, 89]. Proposed by E. J. OGLESBY, Washington Square College.

Given the values of $U_{5:9}$, $U_{5:10}$, $U_{5:11}$, $U_{6:9}$, $U_{6:10}$, $U_{6:11}$, $U_{7:9}$, $U_{7:10}$, $U_{7:11}$ where $U_{h:k} = \sqrt{hk}$, find the value of $U_{6.2:9.3}$ by interpolation.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

From the first three given terms we find by the method of differences[2] $U_{5:9.3} = 6.8190$, from the next three $U_{6:9.3} = 7.4698$ and from the last three $U_{7:9.3} = 8.0685$. Using these three values then as a series by the same method we find $U_{6.2:9.3} = 7.5937$.

Also solved by H. N. CARLETON.

2881 [1921, 89]. Proposed by E. B. ESCOTT, Oak Park, Ill.

If, in the polynomial $X^3 - 2$, we substitute $x^2 + x - 4$ for X, the given expression can be factored, that is, $X^3 - 2 \equiv (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6)$. Find a substitution for X so that the polynomial $X^3 + pX^2 + qX + r$ may be factored.

SOLUTION BY THE PROPOSER.

Let

$$X^3 + pX^2 + qX + r = (X - b)(X + c)^2 - a^2(X - d)^2. \qquad (1)$$

Expanding the second member and equating coefficients of like powers of X, we get

$$a^2 + b - 2c + p = 0, \qquad 2a^2d - 2bc + c^2 - q = 0, \qquad \text{and} \qquad a^2d^2 + bc^2 + r = 0.$$

Eliminating b and d and solving for a^2 we get

$$a^2 = \frac{(3c^2 - 2pc + q)^2}{4(c^3 - pc^2 + qc - r)} .$$

Therefore, it is necessary and sufficient that [3]

$$c^3 - pc^2 + qc - r = n^2,$$

and we get by substitution

$$a = \frac{3c^2 - 2pc + q}{2n}, \qquad d = - \frac{c^3 - qc + 2r}{3c^2 - 2pc + q}, \qquad \text{and} \qquad b = - a^2 + 2c - p.$$

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[1] We might say

$$\frac{1}{n^2} = \frac{\sin^2 \frac{1}{2}a \cos^2 x}{\sin^2 \frac{1}{2}(a + d)} + \frac{\cos^2 \frac{1}{2}a \sin^2 x}{\cos^2 \frac{1}{2}(a + d)}$$

$$= \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)} + \left[\frac{\cos^2 \frac{1}{2}a}{\cos^2 \frac{1}{2}(a + d)} - \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)} \right] \sin^2 x$$

$$= \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)} + \frac{4 \sin \frac{1}{2}d \sin (a + \frac{1}{2}d) \sin^2 x}{\sin^2 (a + d)}$$

and use this formula for (b) and (c) instead of the formula in the text, d being a minimum when $[\sin^2 \frac{1}{2}a]/[\sin^2 \frac{1}{2}(a + d)]$ is a maximum.—EDITORS.

[2] See *1921*, 330.

[3] Thus it seems to be a necessary part of the hypothesis that there is a rational number $- c$ that will make the given polynomial equal to minus the square of a rational number.

Letting f denote the polynomial, if $f(- c) = - n^2$ we can write

$$f(X) = (X + c)^2 \left[X - 2c + p + \left(\frac{f'(-c)}{2n} \right)^2 \right] - \left[\frac{f'(-c)}{2n}(X + c) - n \right]^2,$$

which we can make the difference of two squares by putting $X - 2c + p + \left(\frac{f'(-c)}{2n} \right)^2 = (x + t)^2$.

—EDITORS.

Thus, a, b, c, and d are known. The right-hand member of (1) will be the difference of two squares, and may be factored if we put $X - b = (x + t)^2$; i.e., $X = (x + t)^2 + b$.

In the example given, $p = q = 0$, $r = -2$. The value $c = -1$ gives $n = 1$; whence, $a = 3/2$, $d = 5/3$, $b = -17/4$. $X^3 - 2 = (X + \frac{17}{4})(X - 1)^2 - \frac{1}{4}(3X - 5)^2$. Letting $X + \frac{17}{4}$ $= (x + \frac{1}{2})^2$; i.e., $X = x^2 + x - 4$, and substituting, we have
$X^3 - 2 = [(x + \frac{1}{2})(x^2 + x - 5) + \frac{1}{2}(3x^2 + 3x - 17)][(x + \frac{1}{2})(x^2 + x - 5) - \frac{1}{2}(3x^2 + 3x - 17)]$
$= (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6)$.

2885 [1921, 139].
If A, B, C, X, Y are given collinear points, construct Z so that $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, where $\{ABCX\}$ denotes the cross-ratio of the points A, B, C, X. [From *the Math. Tripos Exam.*, Cambridge, Eng., 1905.]

I. Solution by Gertrude I. McCain, Westminster College, New Wilmington, Pa.

If $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, then $\dfrac{AB}{BC} : \dfrac{AX}{XC} + \dfrac{AB}{BC} : \dfrac{AY}{YC} = \dfrac{AB}{BC} : \dfrac{AZ}{ZC}$. Dividing by $\dfrac{AB}{BC}$, $\dfrac{XC}{AX} + \dfrac{YC}{AY} = \dfrac{ZC}{AZ}$. But $XC = -CX = -(AX - AC)$, $YC = -CY = -(AY - AC)$, and $ZC = -CZ = -(AZ - AC)$. Making the substitutions and adding 2 to each side of the equation, we have $\dfrac{AC}{AX} + \dfrac{AC}{AY} = \dfrac{AC}{AZ} + 1$, or $\dfrac{1}{AX} + \dfrac{1}{AY} = \dfrac{1}{AZ} + \dfrac{1}{AC}$. Let $AX = \cot \theta$, $AY = \cot \varphi$, $AC = \cot \psi$, and $AZ = \cot \tau$. Considering AX, AY and AC as line functions, construct the angles θ, φ and ψ, using a circle with radius unity. Also construct the tangents of θ, φ, and ψ as line functions using the same circle.

It is now possible to lay off $\tan \theta$ and $\tan \varphi$ as lengths on a line and from their sum subtract $\tan \psi$.

But $\dfrac{1}{AX} + \dfrac{1}{AY} - \dfrac{1}{AC} = \dfrac{1}{AZ} = \tan \tau$. Cot τ or AZ may then be constructed as a line function, and the point Z located on the line with A, B, C, X, Y.

II. Solution by Otto Dunkel, Washington University.

After showing that $1/AX + 1/AY = 1/AZ + 1/AC$, it should be observed that this result shows that the two sets of points A, X, Y and A, Z, C have a common fourth harmonic point K. Hence K can be constructed from the first set and then Z as the harmonic conjugate of C with respect to A and K, by the quadrilateral construction, thus giving a non-metrical construction. This would then give a projective definition of the sum of two cross-ratios. This result may be obtained by use of abridged notation as follows:
If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\xi = 0$, $\eta = 0$ are the equations of the points A, B, C, X, Y, then the expressions to the left in these equations may be chosen so that
$$\beta = b\alpha + \gamma, \qquad \xi = x\alpha + \gamma, \qquad \eta = y\alpha + \gamma,$$
and then $\{ABCX\} = x/b$, $\{ABCY\} = y/b$. It follows that $(x - y)\alpha = \xi - \eta$, and if K is the harmonic conjugate of A with respect to X, Y, its equation may be written $(x - y)\kappa = \xi + \eta = 0$ or $(x - y)\kappa = (x + y)\alpha + 2\gamma$. Since $2\gamma = (x - y)\kappa - (x + y)\alpha$, Z, the harmonic conjugate of C with respect to K, A, may have its equation written $2\zeta = (x - y)\kappa + (x + y)\alpha = 0$ or $\zeta = (x + y)\alpha + \gamma$. Hence $\{ABCZ\} = (x + y)/b = \{ABCX\} + \{ABCY\}$.

Also solved by Nathan Altshiller-Court, J. W. Clawson, Arthur Pelletier, and F. L. Wilmer.

2887 [1921, 139]. Proposed by the late L. G. WELD.
A carpenter's square moves with its outer edges in contact with two round pegs of given equal diameters. Define the locus of the "heel" of the square.

Solution by J. B. Reynolds, Lehigh University.

Let the radius of either peg be r and the distance between their centers $\sqrt{2}a$. Let the axes be taken so that the centers of the circles are at the points $(a, 0)$ and $(0, a)$, and assume that the